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NOTE

Factoring, into Edge Transpositions of a Tree, Permutations Fixing a Terminal Vertex

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of a spanning tree we discuss identities they satisfy, including a set of defining relations. We further show that a minimal length factorization of a permutation fixing a terminal vertex does not involve the unique edge incident to that vertex.

If T is a spanning tree on a set X of vertices, and to each edge a of T we associate the transposition, on X, which just interchanges the ends of a, then the resulting transpositions generate S_X. We investigate what relations they satisfy. It should cause no confusion to identify an edge with its transposition in statements such as "edges commute." While it will usually not matter, is is convenient to have our permutations act on the right so in a product the left factor acts first.

The author thanks T. Vaughan for a simplification of the proof of Proposition 6.

PROPOSITION 1. 1. *edges with no common vertex commute*

2. *if a, b, and $c_1, ..., c_k$ are distinct edges with a common vertex and C is a word in the c_i, then aCa commutes with b.*

3. *if $a_1, ..., a_k$ are distinct edges with a common vertex, then $a_1 \cdots a_k a_1 = a_2 \cdots a_k a_1 a_2$.*

4. *If $a_1, ..., a_k$ are distinct edges with a common vertex, then $(a_1 \cdots a_k)^{k+1} = 1$.*

Proof. 1 and 2 are clear, as are cases $k=1$, $k=2$ of 4 from which the case $k=2$ of 3 follows. To prove 3 in for general k insert two a_1 just before

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the a_k, apply the case $k-1$ of 3, and 2. To prove 4 for $k>2$, regroup the $k+1$ groups of k into k groups of $k+1$ and apply 3.

Note that the above argument actually shows:

PROPOSITION 2. *Of the above,* 1, 2 *and cases* $k=1$ *and* $k=2$ *of* 4 $(a^2=(ab)^3=1)$ *imply the rest.*

The next propositions follow formally from the above relations, hence hold in the group defined by them, not only in S_X. In Theorem 8 we will show, essentially, that the distinction does not matter. The proofs of Propositions 3–5 are routine.

PROPOSITION 3. *Let a_1 and a_2 have a common vertex and let B be a word in edges which are joined to a_1 and a_2 only through this common vertex. Then $a_1 B a_1$ commutes with a_2.*

PROPOSITION 4. *Let a_1, a_2 and B be as in Proposition* 3. *Then $a_1 a_2 B a_1 = a_2 B a_1 a_2$*

PROPOSITION 5. *Let a_1, a_2 be distinct edges with a common vertex and suppose the edges of words B_1, (B_2) are connected to this vertex by paths disjoint from B_2, (B_1) and $\{a_1, a_2\}$. Then $a_1 B_1 a_2 B_2 a_1 = a_2 B_2 a_1 B_1 a_2$*

PROPOSITION 6. *Let a_1 be a terminal edge. Then any word $a_1 W$, where W does not contain a_1, is equivalent to one of the form $U a_1 V$ where a_1 does not occur in U or V, V is no longer than W, and $a_1 V = a_1 a_2 \cdots a_k$ is a path.*

Proof. It will suffice to show that if $a_1 W$ is not a path, then some edge of W may be moved to the left of the a_1, possibly introducing edges other than a_1 to the left of the a_1 but without affecting the rest of W.

Let $a_2 \cdots a_k b$ be an initial word of W such that $a_1 \cdots a_k$ is a path but $a_1 \cdots a_k b$ is not. If b has no vertices in common with any of the a_i we may simply take it to the left of all of them. If $b = a_k$ it may simply be cancelled, if it is one of the a_i, $i=1, \cdots k-1$ then

$$\begin{aligned} a_1 \cdots a_k b &= a_1 \cdots a_k a_i \\ &= a_1 \cdots a_{i-1} a_i a_{i+1} a_i a_{i+2} \cdots a_k \\ &= a_1 \cdots a_{i-1} a_{i+1} a_i a_{i+1} a_{i+2} \cdots a_k \\ &= a_{i+1} a_1 \cdots a_k \end{aligned}$$

If b is not one of the a_i but is adjacent to a_1 it is adjacent to a_2 since a_1 is terminal, if it is adjacent to a_k it is to a_{k-1} since $a_1 \cdots a_k b$ is not a path. Suppose b is adjacent to a_j and a_{j-1}. Then

$$\begin{aligned} & a_1 \cdots a_{j-2} a_{j-1} a_j \cdots a_k b \\ & \quad = a_1 \cdots a_{j-2} a_{j-1} a_j b \cdots a_k \qquad \text{by 1.1} \\ & \quad = a_1 \cdots a_{j-2} a_{j-1} (a_j b a_j) a_j a_{j+1} \cdots a_k \\ & \quad = (a_j b a_j) a_1 \cdots a_{j-2} a_{j-1} a_j a_{j+1} \cdots a_k \end{aligned}$$

PROPOSITION 7. *Let a_1 be a terminal edge. Then any word is equivalent to one of the form U or $Ua_1 a_2 \cdots a_k$, where a_1 does not occur in U and $a_1 a_2 \cdots a_k$ is a path.*

Proof. If the word contains more than one a_1, then use Proposition 6 to reduce the number to 0 or 1 and in the latter case apply Proposition 6 one more time.

Proposition 1 shows that there is a homomorphism from the group defined by the above relations to S_X taking each edge to the corresponding transposition.

THEOREM 8. *This homomorphism is an isomorphism.*

Proof. We know it is onto, so it will suffice to show that the relations force the group to have order at most $n!$. Since this is trivial if $n = 2$ we argue by induction. Take v to be a terminal vertex, incident to edge a_1. Then Proposition 7 shows that any word W is equivalent to a word VP, where V is one of $(n-1)!$ possibilities (by induction) and P is one of n possible paths (including the path of 0 length).

The above means that any identity we can prove regarding the edges as transpositions and computing in the symmetric group follows formally from the identities of Proposition 1.

An expression of a permutation as a product of edges of T will be called a T-factorization, or just factorization if the T is clear. The length of a factorization is simply the number of edge transpositions, and a factorization of a permutation is minimal if there is no shorter factorization of the same permutation.

THEOREM 9. *Suppose v is a vertex of degree 1 in a tree T and suppose that the permutation σ fixes v. Then no minimal length T-factorization of σ contains the edge incident with v.*

Proof. Suppose not. Take a counterexample consisting of a permutation σ with a fixed terminal vertex v and a minimal factorization involving the edge a_1 incident to v. We may assume the example is minimal in the sense that no τ having a fixed terminal vertex has a shorter T-factorization involving the edge incident to that vertex. Then in particular the word begins and ends with a_1.

We may assume it has an initial word of the form $a_1 W a_1$ where a_1 does not occur in W. By Proposition 6 we may assume that this word is of the form $a_1 a_2 \cdots a_k a_1$ where $a_1 a_2 \cdots a_k$ is a path. But we may move the a_1 on the right to the left to get $a_1 a_2 a_1 \cdots a_k = a_2 a_1 a_2 \cdots a_k$ and removing the a_2 yields a shorter example.

The above theorem answers Question 2 of [3]. The case $k=0$ of Question 4, is:

Conjecture. If each of the subtrees into which an edge separates T is setwise invariant under σ, then some minimal factorization of σ does not use the edge.

(A simple example showing that the answer for $k>0$ is negative has vertices $\{1, 2, 3, 4, 5, 6\}$, $T=\{a=(13), b=(35), c=(34), d=(24), e=(46)\}$, $\sigma=(162)(45)(3)$. Here σ only switches one pair across a, but the only minimal length factorizations, *acbdeca*, *acdbeca*, and *acdebca* use a twice.)

Perhaps

Conjecture. If each of the subtrees into which an edge separates T is setwise invariant under σ, then no minimal factorization of σ uses the edge.

(Theorem 9 is the case where one of the subtrees is a single vertex).

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